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Real Analysis I

Problems

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Abstract: Here's a list of practice problems organized by topic for the UT Austin preliminary examination on Real Analysis (i.e. the Analysis I exam). They have mostly come from the old prelim exam page, with some additional exercises dispersed throughout.

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1 Measure Theory

- (1) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz transformation. Show that if A is a set of Lebesgue measure zero, then T(A) also has Lebesgue measure zero.
- (2) For any $r \ge 0$ and any $x \in \mathbb{R}^2$, define the closed unit ball

$$B_r(x) := \{ y \in \mathbb{R}^2 : |y - x| \le r \}.$$

Let 0 < c < 1. Let E be a measurable subset of the unit square $Q = [0, 1]^2 \subset \mathbb{R}^2$ with the property that for every $x \in Q$ and every r > 0 there exists a $y \in B_r(x)$ such that $B_{c|x-y|}(y) \subset E$. Prove that $Q \setminus E$ has Lebesgue measure zero.

(3) Let (X, d) be a compact metric space. Let $\{\mu_n\}_{n \in \mathbb{N}}$ be a sequence of positive Borel measures on X that converge in the weak* topology to a finite positive Borel measure μ . Show that for every compact $K \subset X$,

$$\mu(K) \ge \limsup_{n \to \infty} \mu_n(K).$$

- (4) Let Z be a subset of R with measure zero. Show that the set $A = \{x^2 : x \in Z\}$ also has measure zero.
- (5) Let $E \subset \mathbb{R}$ be a measurable set such that $0 < |E| < \infty$. Prove that for every $\alpha \in (0, 1)$ there is an open interval I such that

$$|E \cap I| \ge \alpha I.$$

(6) Assume that μ is a finite Borel measure on \mathbb{R}^n , and that there exists a constant $0 < R < \infty$ such that the k-th moments of μ satisfy the bound

$$\int |x|^k d\mu < R^{k^r} \qquad \forall k \in \mathbb{N},$$

for some $0 < r \le 1$. Prove that μ has bounded support contained in $\{x \in \mathbb{R}^n : |x| \le R\}$ if r = 1 and in $\{x \in \mathbb{R}^n : |x| \le 1\}$ if 0 < r < 1.

(7) Let μ be a measure in the plane for which all open squares are measurable, with the property that there exists $\alpha \geq 1$ such that if two open squares Q, Q' are translates of each other and their closures $\overline{Q}, \overline{Q'}$ have a non-empty intersection, then

$$\mu(\overline{Q}) \le \alpha \mu(Q') < \infty.$$

(For Lebesgue measure, $\alpha = 1$. In general $\alpha \ge 1$.) Show that horizontal lines have zero μ -measure.

- (8) Let μ be a Borel measure on [0, 1]. Assume that
 - (a) μ and Lebesgue measure are mutually singular.
 - (b) $\mu([0, t])$ depends continuously on t.
 - (c) For any function $f : [0,1] \to \mathbb{R}$, if $f \in L^1(\text{Lebesgue})$ then $f \in L^1(\mu)$. (Note that f is finite valued)

Show that $\mu \equiv 0$.

- (9) Show that the following notions of measurability are equivalent. Here, we let $\lambda : 2^{\mathbb{R}} \to [0, \infty]$ be the Lebesgue outer measure.
 - (a) $E \subset \mathbb{R}$ is measurable iff for every $\epsilon > 0$ there exists an open set $O \supset E$ such that $\lambda(O \setminus E) < \epsilon$.
 - (b) $E \subset \mathbb{R}$ is measurable iff for every set $A \subset \mathbb{R}$ (measurable or not) we have

$$\lambda(A \cap E) + \lambda(A \cap E^c) = \lambda(E).$$

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2 Convergence in Measure

- (1) Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of measurable functions on \mathbb{R}^n that converge to zero in measure on $\{y \in \mathbb{R}^n : |y - x| \leq 1\}$ and are uniformly bounded in $L^2(\mathbb{R}^n)$. Show that f_k converges to zero in $L^1(\mathbb{R}^n)$.
- (2) Prove that, on a finite measure space, if $f_k \to f$ in measure and $g_k \to g$ in measure, then $f_k g_k \to f g$ in measure.
- (3) Suppose that $f_k \to f$ in measure on \mathbb{R} with respect to Lebesgue measure and $|f_k| \leq g \in L^1(\mathbb{R})$ for all k. Prove that $f_k \to f$ in $L^1(\mathbb{R})$.
- (4) Let (X, Σ, μ) be a finite measure space and $1 \leq q . Let <math>\{f_k\}_{k=1}^{\infty} \subset L^p(X, \mu)$ and $\|f_k\|_p \leq 1$ for all k. Assuming that $f_k \to f$ in measure, show that $f \in L^p(X, \mu)$ and that $\|f_k f\|_q \to 0$.

3 Density and Limits in L^p spaces

1 Show that $\mathcal{C}_c(\mathbb{R}^n) := \{ f \in \mathcal{C}(\mathbb{R}^n) : f \text{ has compact support} \}$ is dense in $L^1(\mathbb{R}^n)$.

(2) Find an uncountable family of measurable functions

$$\mathcal{F} \subset \{ f : \mathbb{R} \to \mathbb{R} \text{ measurable } \}$$
(1)

that satisfies the following two conditions:

- (a) For all $f \in \mathcal{F}$, $||f||_{L^{\infty}(\mathbb{R})} = 1$
- (b) For all $f, g \in \mathcal{F}$, we have $||f g||_{L^{\infty}(\mathbb{R})} = 1$.

(Bonus: Show that this implies L^{∞} is not separable.)

- (3) Let $1 \le p, q \le \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Show that if $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ then f * g is bounded and continuous on \mathbb{R}^n .
- (4) Let B be the closed unit ball in \mathbb{R}^n , and let $\{f_k\}_{k\in\mathbb{N}}$ be nonnegative integrable functions on B. Assume that
 - (a) $f_k \to f$ almost everywhere.
 - (b) For every $\epsilon > 0$ there exists an M > 0 such that

$$\int_{\{x\in B: f_k(x)>M\}} f_k(x) dx < \epsilon \qquad \forall k \in \mathbb{N}.$$

Show that $f_k \to f$ in $L^1(B)$.

- (5) Let $\{f_k\}_{k\in\mathbb{N}} \subset L^p$ with $1 \le p < \infty$. If $f_k \to f$ pointwise a.e. and $||f_k||_p \to ||f||_p$, show that $||f f_k||_p \to 0$.
- (6) Let $f \in L^1(\mathbb{R})$ and φ_{ϵ} be a mollifier. This measn that $\varphi_{\epsilon}(x) = \epsilon^{-1}\varphi(x/\epsilon)$ and $\varphi : \mathbb{R} \to \mathbb{R}$ is a function that satisfies: $\varphi \ge 0$, φ is compactly supported, and $\int \varphi = 1$. Let $f_{\epsilon} := f * \varphi_{\epsilon}$. Show that

$$\int_{\mathbb{R}} \liminf_{\epsilon \to 0} |f_{\epsilon}| \le \int_{\mathbb{R}} |f|.$$

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(7) Let $f \in L^1(X,\mu)$. Prove that for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left|\int_{A} f d\mu\right| < \epsilon$$

for all measurable $A \subset X$ such that $\mu(A) < \delta$.

- (8) Let $p \in [1, \infty)$ and suppose $\{f_n\}_{n=1}^{\infty} \subset L^p(\mathbb{R})$ is a sequence that converges to 0 in $L^p(\mathbb{R})$. Prove that one can find a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to 0$ almost everywhere.
- (9) Show that, if $f \in L^4(\mathbb{R})$, then

$$\int |f(\lambda x) - f(x)|^4 dx \to 0$$

as $\lambda \to 1$.

- (10) Let f, g be bounded measurable functions on \mathbb{R}^n . Assume that g is integrable and satisfies $\int g = 0$. Define $g_k(x) = k^n g(kx)$ for $k \in \mathbb{N}$. Show that $f * g_k \to 0$ pointwise a.e. as $k \to \infty$.
- (11) Suppose that $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of integrable functions on [0,1] such that $\|f_n\|_{L^1([0,1])} \leq n^{-2}$ for all $n \in \mathbb{N}$. Show that $f_n \to 0$ pointwise a.e.
- (12) Let $f \in L^{\infty}(\mu)$ be a nonnegative bounded μ -measurable function. Consider the set R_f consisting of all positive real numbers w such that $\mu(\{x : |f(x) w| \le \epsilon\}) > 0$ for every $\epsilon > 0$.
 - (a) Prove that R_f is compact.
 - (b) Prove that $||f||_{L^{\infty}} = \sup R_f$.
- (13) Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of $L^1([0,1])$ functions such that $f_k \to f$ pointwise a.e. Show that $||f_k - f||_{L^1([0,1])} \to 0$ if and only if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\left|\int_{A} f_k\right| < \epsilon$$

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for all k and all measurable sets $A \subset [0, 1]$ with measure $|A| < \delta$.

- (14) Let $f_k \to f$ a.e. on \mathbb{R} . Show that given $\epsilon > 0$, there exists E with $|E| < \epsilon$ so that $f_k \to f$ uniformly on $I \setminus E$ for any finite interval I.
- (15) Let (X, \mathcal{A}, μ) be a measure space with $\mu(X) < \infty$. Show that a measurable function $f: X \to [0, \infty)$ is integrable if and only if $\sum_{n=0}^{\infty} \mu(\{x \in X : f(x) \ge n\})$ converges.
- (16) Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and $f \in L^1(\Omega)$. Prove that

$$\lim_{p \to 0} \left[\int_{\Omega} |f|^p d\mu \right]^{1/p} = \exp\left[\int_{\Omega} \log |f| d\mu \right]$$

where $\exp[-\infty] = 0$. To simplify the problem, you may assume that $\log |f| \in L^1(\Omega)$.

(17) Let h be a bounded, measurable function such that for any interval I

$$\left| \int_{I} h \right| \le |I|^{1/2}.$$

Let $h_{\epsilon}(x) = h(x/\epsilon)$. Show that for any A with $|A| < \infty$

$$\int_A h_\epsilon \to 0, \qquad \text{as } \epsilon \to 0$$

(18) For $\frac{1}{p} + \frac{1}{q} = 1$, let $S = \{ f \in L^p(\mathbb{R}) : \text{ support}(f) \subset [-1, 1], \text{ and } \|f\|_{L^p} \le 1 \}$

and let g be a fixed but arbitrary function in $L^q(\mathbb{R})$ with $\operatorname{support}(g) \subset [-1, 1]$. Show that the image of S under the map $f \mapsto f * g$ is a compact set in $\mathcal{C}([-2, 2])$.

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(19) Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence on non-negative Lebesgue integrable functions on \mathbb{R}^n such that

$$\sum_{k=1}^{\infty} \int (f_k - f_{k-1})_+ < \infty, \quad \text{and} \quad \lim_{k \to \infty} \int f_k = 0.$$

Show that $\limsup_{k\to\infty} f_k = 0$ a.e.

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4 Weak L^p and Fubini

(1) Let H be a monotone function of f, a non-negative measurable function. Write

$$\int H(f(x))dx$$

in terms of $g(\lambda) := |\{f > \lambda\}|.$

(2) Show that if p > 1 and $f \in L^p([0,\infty))$, then the mean functional of f, defined by

$$F(y) := \frac{1}{y} \int_0^y f(t) dt = \int_0^1 f(xy) dx$$

is also in $L^p([0,\infty))$ and moreover

$$||F||_p \le \frac{p}{p-1} ||f||_p.$$

(Hint: Use the generalized Minkowski inequality).

- (3) Let f be a locally integrable function on \mathbb{R}^2 . Assume that, for any given real numbers a and b outside of some set of measure zero, f(x, a) = f(x, b) for almost every $x \in \mathbb{R}$ and f(a, y) = f(b, y) for almost every $y \in \mathbb{R}$. Show that f is constant almost everywhere on \mathbb{R}^2 .
- (4) Let f, g be real valued measurable integrable functions on a measure space (X, μ) and let

$$F_t := \{ x \in X : f(x) > t \}, \qquad G_t := \{ x \in X : G(x) > t \}.$$

Prove that

$$\|f - g\|_1 = \int_{-\infty}^{\infty} \mu(F_t \Delta G_t) dt$$

where $F_t \Delta G_t = (F_t \setminus G_t) \cup (G_t \setminus F_t)$.

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(5) Let $0 < q < p < \infty$. Let $E \subset \mathbb{R}^n$ be measurable with measure $|E| < \infty$. Let f be a measurable function on \mathbb{R}^n such that

$$N := \sup_{\lambda > 0} \lambda^p |\{ x \in \mathbb{R}^n : |f(x)| > \lambda \}|$$

is finite.

- (a) Prove that $\int_E |f|^q$ is finite.
- (b) Refine the argument of (a) to prove that

$$\int_E |f|^q \le CN^{q/p} |E|^{1-q/p}$$

where C is a constant depending only on n, p, and q.

- (6) Let p > 0. Give the definition of $L^p_{weak}(\mathbb{R})$ and the quasi-norm N_p on this space. Prove that simple functions are *not* dense in $L^p_{weak}(\mathbb{R})$. That is, there exists an $f \in L^p_{weak}(\mathbb{R})$ such that $N_p(f - h_k) \to 0$ fails to hold for every sequence of simple functions $\{h_k\}_{k=1}^{\infty}$.
- (7) Let $1 and <math>f(x) = |x|^{-n/p}$ for $x \in \mathbb{R}^n$. Prove that $f \in L^p_{weak}(\mathbb{R}^n)$ and f is not the limit of a sequence of $\mathcal{C}^{\infty}_c(\mathbb{R}^n)$ functions.

5 Maximal Functions

(1) For $f \in L^1(\mathbb{R})$ denote by Mf the restricted maximal function defined by

$$(Mf)(x) := \sup_{0 < t < 1} \frac{1}{2t} \int_{x-t}^{x+t} |f(z)| dz.$$

Show that $M(f * g) \leq (Mf) * (Mg)$ for all $f, g \in L^1(\mathbb{R})$.

(2) For a function $f \in L^1(\mathbb{R}^2)$ let $\widetilde{M}f$ be the unrestricted maximal function

$$\widetilde{M}f(x_0, y_0) := \sup_Q \frac{1}{|Q|} \int_Q |f(x, y)| dxdy$$

where the supremum is over all closed cubes $Q \subset \mathbb{R}^2$ centered at (x_0, y_0) .

(a) Show that $\widetilde{M}f(x_0, y_0) \leq M_1 M_2 f(x_0, y_0)$, where

$$M_1f(x_0,y) := \sup_{k>0} \frac{1}{2k} \int_{x_0-k}^{x_0+k} |f(x,y)| dx, \qquad M_2f(x,y_0) = \sup_{l>0} \frac{1}{2l} \int_{y_0-l}^{y_0+l} |f(x,y)| dy.$$

(b) Show that there exists a constant C > 0 such that if $f \in L^2(\mathbb{R}^2)$ then

$$\|\widetilde{M}f\|_{L^2(\mathbb{R}^2)} \le C \|f\|_{L^2(\mathbb{R}^2)}.$$

(3) Consider the Hardy-Littlewood maximal function (for balls)

$$Mf(x) := \sup_{B} \frac{1}{|B|} \int_{B} |f|, \qquad f(x) := \begin{cases} 1, & \text{if } |x| \le 1\\ 0, & \text{if } |x| > 1 \end{cases}$$

Prove that Mf belongs to $L^1_{weak}(\mathbb{R}^n)$.

6 Weak Derivatives and Absolute Continuity

- (1) Let $f, \{f_k\}_{k \in \mathbb{N}}$ be increasing functions on [a, b]. If $\sum_{k=1}^{\infty} f_k$ converges pointwise to f on [a, b], show that $\sum_k f'_k$ converges to f' almost everywhere on [a, b].
- (2) Let $1 . Assume <math>f \in L^p(\mathbb{R})$ satisfies

$$\sup_{0<|h|<1}\int \left|\frac{f(x+h)-f(x)}{h}\right|^p < \infty.$$

Show that f has a weak derivative $g \in L^p$, which by definition satisfies $\int \psi g = -\int \psi' f$ for every $\psi \in \mathcal{C}^{\infty}_{c}(\mathbb{R})$.

- (3) Assuming $f : [0,1] \to \mathbb{R}$ is absolutely continuous, prove that f is Lipschitz if and only if f' belongs to $L^{\infty}([0,1])$.
- (4) Let f be a non-decreasing function on [0, 1]. You may assume that f is differentiable a.e.
 - (a) Prove that

$$\int_0^1 f'(t)dt \le f(1) - f(0).$$

- (b) Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of non-decreasing functions on [0,1] such that $F(x) = \sum_{n=1}^{\infty} f_n(x)$ converges for $x \in [a,b]$. Prove that $F'(x) = \sum_{n=1}^{\infty} f'_n(x)$.
- (5) Is the function $f:[0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x \sin(1/x), & x > 0\\ 0, & x = 0 \end{cases}$$

absolutely continuous on [0, 1]? Explain fully.

(6) Let $f : \mathbb{R} \to \mathbb{R}$ be absolutely continuous with compact support, and let $g \in L^1(\mathbb{R})$. Prove that f * g is absolutely continuous on \mathbb{R} .

7 Explicit Computations and Counterexamples

- (1) Find a non-empty closed set in $L^2([0,1])$ which does not contain an element of minimal norm.
- (2) Give an example of a sequence $\{f_n\}_{n\in\mathbb{N}}\subset L^1(\mathbb{R})$ such that $f_n\to 0$ a.e. on \mathbb{R} , but f_n does not converge to 0 in $L^1_{loc}(\mathbb{R})$.
- (3) For any natural number n, construct a function $f \in L^1(\mathbb{R}^n)$ such that for any ball $B \subset \mathbb{R}^n$, f is not essentially bounded on B.
- (4) Let $g \in L^1(\mathbb{R}^n)$ with $||g||_{L^1(\mathbb{R}^n)} < 1$. Prove that there is a unique $f \in L^1(\mathbb{R}^n)$ such that

$$f(x) + (f * g)(x) = e^{-|x|^2}, \qquad x \in \mathbb{R}^n \text{ a.e.}$$

- (5) Provide an example of a sequence of measurable functions on [0, 1] which converges in L^1 to the zero function but does not converge pointwise a.e.
- (6) Let $(x_n)_{n \in \mathbb{N}}$ be an arbitrary sequence of real numbers in [0, 1] (possibly dense). Show that the series $\sum_k k^{-3/2} |x - x_k|^{-1/2}$ converges for almost every $x \in [0, 1]$.
- (7) Let f be a continuous function on [0, 1]. Find

$$\lim_{n \to \infty} n \int_0^1 x^n f(x) dx.$$

Justify your answer.

(8) If $f(x,y) \in L^2(\mathbb{R}^2)$, show that $f(x+x^3, y+y^3) \in L^1(\mathbb{R}^2)$.