Real Analysis I

Lecture Notes

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Abstract: These notes were prepared for a summer "bootcamp" to prepare graduate students in mathematics for the real analysis prelim exam. The sections, theorems, and definitions marked with $*$ below will stand for more advanced topics which are less likely to appear on exams.

Contents

1 Measure Theory

Definition 1 (Outer Measure)**.** *Let X be a set and $\mu : 2^X \to [0,\infty]$. We say μ is an outer measure if it is countably sub-additive and $\mu(\emptyset) = 0$.*

Definition 2 (Lebesgue Measure)**.** *The Lebesgue measure $\lambda_d : 2^{\mathbb{R}^d} \to [0,\infty]$ is defined by*

$$\lambda_d(E) := \inf \left\{ \sum_{k=1}^{\infty} vol(Q_k) : \quad E \subset \bigcup_{k=1}^{\infty} Q_k, \quad Q_k \ cubes \right\} \tag{1}$$

where $vol(Q)$ is the volume of the cube Q. It is an outer measure.

Definition 3 (Measure)**.** *Let X be a set and $\mathcal{F} \subset 2^X$ be a σ-algebra. A function $\mu : \mathcal{F} \to [0,\infty]$ is a measure if μ is countably additive. The sets $E \in \mathcal{F}$ are the measurable sets.*

Theorem 1 (Carethéodory's Theorem)**.** *Let X be a set and μ be an outer measure. Then the set*

$$\mathcal{F} := \{E \subset X : \forall A \subset X, \quad \mu(A) = \mu(A \cap E) + \mu(A \cap E^c)\} \tag{2}$$

is a σ-algebra, and $\mu : \mathcal{F} \to [0,\infty]$ is a measure. We therefore say $E \in \mathcal{F}$ is measurable with respect to the outer measure μ.

Theorem 2 (Lebesgue Measurability)**.** *For each $E \subset \mathbb{R}^d$, the following are equivalent*

- *For all $A \subset \mathbb{R}^d$, $\lambda_d(A) = \lambda_d(A \cap E) + \lambda_d(A \cap E^c)$.*

- *For each $\epsilon > 0$, there exists an open set $O \supset A$ such that $\lambda_d(O \setminus A) < \epsilon$.*

Theorem 3 (Measuring Tools)**.** *Let μ be a measure on X. Let A_k be measurable for $k \in \mathbb{N}$.*

- *If $A_1 \subset A_2 \subset \cdots$ and $A^* = \bigcup_{k=1}^{\infty} A_k$, then $\mu(A^*) = \lim_{k\to\infty} \mu(A_k)$.*

- *If $A_1 \supset A_2 \supset \cdots$, $A_* = \bigcap_{k=1}^{\infty} A_k$, and $\mu(A_1) < \infty$ then $\mu(A_*) = \lim_{k\to\infty} \mu(A_k)$.*

- *Let us define*
$$\limsup_{k\to\infty} A_k := \bigcap_{k=1}^{\infty}\bigcup_{j=k}^{\infty} A_j.$$
If $\sum_{k=1}^{\infty}\mu(A_k) < \infty$*, then* $\mu(\limsup_{k\to\infty} A_k) = 0$*. This implication is known as the Borel-Cantelli Lemma.*

Definition 4 (Singular and Absolutely Continuous)**.** *Let* μ, ν *be two measures on* X*.*

- *We say* μ *is singular with respect to* ν *(written* $\mu \perp \nu$*) if there exists a measurable set* $Z \subset X$ *such that*
$$\mu(Z) = \nu(X \setminus Z) = 0.$$
- *We say* μ *is absolutely continuous with respect to* ν *(written* $\mu \ll \nu$*) if for all measurable* $A \subset X$*,*
$$\nu(A) = 0 \implies \mu(A) = 0.$$

Theorem 4 (Jordan Decompsition*)**.** *Let* $\mu : 2^X \to [-\infty, \infty]$ *be a signed measure. Then there exists unique positive measures* μ^+, μ^- *such that* $\mu = \mu^+ - \mu^-$*.*

Theorem 5 (Hahn Decomposition*)**.** *Let* $\mu : 2^X \to [-\infty, \infty]$ *be a signed measure. Then there exists measurable sets* $P, N \subset X$ *such that* $P \cap N = \emptyset$ *and* $P \cup N = X$ *with the property that*
$$E \subset P \implies \mu(E) \geq 0$$
$$E \subset N \implies \mu(E) \leq 0.$$
We call P *the positive part and* N *the negative part of* μ*. They satisfy*
$$\mu^+(E) = \mu(E \cap P), \qquad \mu^-(E) = -\mu(E \cap N).$$

Theorem 6 (Lebesgue Decomposition*)**.** *Let* μ, ν *be* σ*-finite signed measures. Then there exists unique signed measures* $\mu_\perp$ *and* $\mu_\ll$ *on* X *such that* $\mu_\perp \perp \nu$*,* $\mu_\ll \ll \nu$*, and*
$$\mu = \mu_\perp + \mu_\ll.$$

Theorem 7 (Radon-Nikodym*)**.** *Let* μ, ν *be two signed* σ*-finite measures on* X*. If* $\mu \ll \nu$*, then there exists a unique measurable function* $f \in L^1(X, d\nu)$ *such that*
$$\mu(E) = \int_E f \, d\nu.$$
We call f *the Radon-Nikodym derivative* $\frac{d\mu}{d\nu}$*.*

Definition 5 (Weak Limits)**.** *Let $\{\mu_n\}_{n\in\mathbb{N}}, \mu$ be finite measures on a metric space X. We say μ_n weakly converges to μ (written $\mu_n \rightharpoonup \mu$) if for every bounded continuous function $\phi : X \to \mathbb{R}$ we have*

$$\int_X \phi d\mu_n \to \int_X \phi d\mu.$$

2 Measurable Functions

Definition 6 (Measurability)**.** *Let $(X, \mathcal{F}, \mu)$ be a measure space. Then $f : X \to \overline{\mathbb{R}}$ is measurable if for all $\lambda \in \mathbb{R}$,*

$$\{x \in X : f(x) \geq \lambda\} \in \mathcal{F}.$$

Theorem 8 (Measurable Function $\approx$ Simple)**.** *Let $(X, \mathcal{F}, \mu)$ be a measure space, and $f : X \to [0, \infty]$ measurable. Then there exists an increasing sequence of simple functions*[1] *S_k such that $f(x) = \lim_{k\to\infty} S_k(x)$ for every $x \in X$.*

Theorem 9 (Egorov)**.** *Let $E \subset \mathbb{R}^n$ be a finite measurable set and $\{f_n\}$ a sequence of measurable functions on E which converge pointwise a.e. to a limit f. Then for all $\epsilon > 0$, there exists a set $F \subset E$ such that $|F| < \epsilon$ and f_n converges uniformly to f on $E \setminus F$.*

Theorem 10 (Lusin)**.** *Let $E \subset \mathbb{R}^n$ be a measurable set with finite measure, and let $f : E \to \mathbb{R}$ be finite valued. Then f is measurable if and only if for every $\epsilon > 0$ there exists a set $F \subset E$ with $|F| < \epsilon$ such that $f : E \setminus F \to \mathbb{R}$ is continuous (with respect to the subspace topology).*

Definition 7 (Convergence in Measure)**.** *Let $\{f_n\}$ and f be measurable functions defined on a measurable set $E \subset \mathbb{R}^n$. Then f_n is said to converge in measure to f if for every $\lambda > 0$ we have*

$$\lim_{k\to} |\{x \in E : |f_n(x) - f(x)| \geq \lambda\}| = 0.$$

3 Integration

Theorem 11 (Chebyshev's Inequality)**.** *Let $f \in L^1(E)$. Then we have for all $\alpha > 0$ that*

$$|\{x \in E : |f(x)| \geq \alpha\}| \leq \frac{1}{\alpha} \int_E |f(x)| dx.$$

[1] A simple function is a finite sum of characteristic functions of measurable sets.

Theorem 12 (Fatou's Lemma). *Let $\{f_n\}_{n=1}^{\infty} \subset L^1(E)$ be a sequence of positive functions. Then we have*

$$\int_E \liminf_{n\to\infty} f_n(x)dx \leq \liminf_{n\to\infty} \int_E f_n(x)dx.$$

This holds even if the left hand side is infinite.

Theorem 13 (Monotone Convergence). *Let $\{f_n\}_{n=1}^{\infty} \subset L^1(E)$ be an* increasing *sequence of positive functions which converge pointwise almost everywhere to f. Then, regardless of the integrability of f, we have*

$$\lim_{n\to\infty} \int_E f_n(x)dx = \int_E f(x)dx.$$

Theorem 14 (Bounded Convergence). *Let $|E| < \infty$. If $\{f_n\} \subset L^1(E)$ is a sequence of functions such that $|f_n| \leq M < \infty$ for all n, and $f_n \to f$ a.e., then*

$$\lim_{n\to\infty} \int_E f_n(x)dx = \int_E f(x)dx.$$

Theorem 15 (Dominated Convergence). *Let $\{f_n\} \subset L^1(\mathbb{R}^d)$ be a sequence of functions which converge pointwise a.e. to $f \in L^1(\mathbb{R}^d)$. If there is an integrable function $g \in L^1(\mathbb{R}^d)$ such that $|f_n| \leq |g|$ for all n, then we have*

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n(x)dx = \int_{\mathbb{R}^d} f(x)dx.$$

Theorem 16 (Jensen's Inequality). *Let $f \in L^1(E)$ with $|E| < \infty$, and let $\varphi : \mathbb{R} \to \mathbb{R}$ be a* convex *function. Then we have*

$$\varphi\left(\frac{1}{|E|}\int_E f(x)dx\right) \leq \frac{1}{|E|}\int_E \varphi(f(x))dx.$$

This inequality is reversed if φ is concave.

Theorem 17 (Generalized Dominated Convergence*). *Let $\{f_n\} \subset L^1(\mathbb{R}^d)$ be a sequence of functions that converge pointwise a.e. to f. Let $\{g_n\} \subset L^1(\mathbb{R}^d)$ be a sequence of functions such that*

- *$|f_n| \leq |g_n|$ for each n,*
- *There exists a $g \in L^1(\mathbb{R}^d)$ such that $g_n \to g$ a.e.*

- *We have as $n \to \infty$*

$$\int_{\mathbb{R}^d} g_n(x)dx \to \int_{\mathbb{R}^d} g(x)dx$$

Then, we obtain that

$$\int_{\mathbb{R}^d} f_n(x)dx \to \int_{\mathbb{R}^d} f(x)dx.$$

4 L^p spaces

Definition 8 (Conjugate Exponents)**.** *Let $1 \leq p \leq \infty$. We define p' to be the* conjugate exponent *to p given by the unique number which satisfies*

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

For $p = 1$, we take $p' = \infty$ and for $p = \infty$, we take $p' = 1$.

Theorem 18 (Hölder's Inequality)**.** *Let $1 \leq p \leq \infty$. If p' is the conjugate exponent to p, then for any f, g measurable we have*

$$\int |fg| \leq \|f\|_{L^p} \|g\|_{L^{p'}}$$

Theorem 19 (Minkowski's Inequality)**.** *Let f, g be measurable functions and $1 \leq p \leq \infty$. Then*

$$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$$

Theorem 20 (Reverse Hölder)**.** *Let f be a measurable function. Then for $1 \leq p \leq \infty$ we can calulate*

$$\|f\|_{L^p} = \sup \left\{ \left| \int_{\mathbb{R}^d} f(x)g(x)dx \right| : \ g \in L^{p'} \text{ and } \|g\|_{L^{p'}} \leq 1 \right\}.$$

Theorem 21 (L^p Duality/Riesz Representation)**.** *Let $1 < p < \infty$. Then*

$$\left(L^p(\mathbb{R}^d)\right)^* \simeq L^{p'}(\mathbb{R}^d).$$

That is, any continuous linear map $T : L^p(\mathbb{R}^d) \to \mathbb{R}$ can be identified with a function $g \in L^{p'}(\mathbb{R}^d)$ such that

$$T(f) = \int_{\mathbb{R}^d} f(x)g(x)dx.$$

Definition 9 (Weak Convergence in L^p)**.** *For $1 < p < \infty$ we say a sequence $\{f_n\} \subset L^p$ converges weakly to $f \in L^p$, written as $f_n \rightharpoonup f$ if for every $g \in L^{p'}$ we have*

$$\int f_n g \to \int fg.$$

For $p = \infty$, we can give an identical definition, but instead we call it weak * convergence *(this is because $(L^1)^* \simeq L^\infty$, but $(L^1)^{**} \not\simeq L^1$).*

Theorem 22 (Weak Compactness in L^p)**.** *For $1 < p < \infty$, for every bounded sequence $\{f_n\} \subset L^p(\mathbb{R}^d)$, there exists a subsequence $\{f_{n_k}\}$ and $f \in L^p(\mathbb{R}^d)$ such that $f_{n_k} \rightharpoonup f$. Equivalently, this says that the unit ball on L^p is compact in the weak topology.*

Definition 10 (Convolution)**.** *For two functions $f, g \in L^1(\mathbb{R}^d)$ we define their convolution by*

$$f * g(x) := \int_{\mathbb{R}^d} f(x-y)g(y)dy.$$

One can check that this defines an L^1 function. This same formula defines convolution on different classes of spaces.

Theorem 23 (L^2 as a Hilbert Space)**.** *We can define an inner product on $L^2(\mathbb{R}^d)$ by considering*

$$\langle f, g\rangle_{L^2} := \int_{\mathbb{R}^d} f(x)\overline{g(x)}dx.$$

Definition 11 (Weak L^p)**.** *A measurable function f lies in weak L^p if*

$$N_p(f) := \sup_{\lambda > 0} \lambda^p |\{x : |f(x)| \geq \lambda\}| < \infty$$

We call the quantity $N_p(f)$ the weak L^p norm of f.

5 Fubini's Theorem

Theorem 24 (Fubini's Theorem)**.** *Let $f = f(x, y) \in L^1(\mathbb{R}^d \times \mathbb{R}^d)$. Then, for almost every x, the function $y \mapsto f(x, y)$ is measurable and in $L^1(\mathbb{R}^d)$. Moreover, we have*

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} f(x, y)d(x, y) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x, y)dxdy = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x, y)dydx.$$

Theorem 25 (Tonelli's Theorem). *Let $f = f(x, y) \geq 0$ be measurable. Then for almost every x the function $y \mapsto f(x, y)$ is measurable and*

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} f(x, y) d(x, y) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x, y) dx dy = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x, y) dy dx.$$

This holds regardless of the integrability of f.

Theorem 26 (Layer Cake Formula). *For $1 \leq p < \infty$, we can calculate the L^p norm of a measurable function f by the formula*

$$\int_{\mathbb{R}^d} |f|^p = p \int_0^\infty \lambda^{p-1} |\{x \in \mathbb{R}^d : |f(x)| \geq \lambda\}| d\lambda$$

6 Lebesgue Differentiation Theorem

Theorem 27 (Lebesgue Differentiation Theorem). *Let $f \in L^1_{loc}(\mathbb{R}^d)$. Then for almost every point $x \in \mathbb{R}^d$, we have*

$$\lim_{r \to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)| dy = |f(x)|.$$

This theorem can be generalized to

$$\lim_{r \to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - f(x)| dy = 0,$$

and for more general classes of balls B_r.

Definition 12 (Maximal Function). *For $f \in L^1(\mathbb{R}^d)$, we define the Hardy-Littlewood Maximal function by*

$$\mathcal{M} f(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy$$

where the supremum is taken over all balls containing x.

Theorem 28 (Vitali's Covering Lemma). *Let $\mathcal{G}$ be a possibly uncountable collection of balls $B \subset \mathbb{R}^d$ which have uniformly bounded diameter. Then there exists a countable collection of disjoint balls $\{B_k\}_{k=1}^\infty$ such that*

$$\bigcup_{\mathcal{G}} B_\alpha \subset \bigcup_{k=1}^\infty 5B_k.$$

Here, we are using the notation that $5B_k$ is the ball with the same center as B_k but with 5 times its radius.

Theorem 29 (Hardy-Littlewood Maximal Inequality)**.** *There exists a constant* $C > 0$ *such that for any* $f \in L^1(\mathbb{R}^d)$ *we have*

$$|\{x \in \mathbb{R}^d : |\mathcal{M}f(x)| \geq \lambda\}| \leq \frac{C}{\lambda}\|f\|_{L^1(\mathbb{R}^d)}.$$

Equivalently, the maximal function $\mathcal{M} : L^1(\mathbb{R}^d) \to L^1_{weak}(\mathbb{R}^d)$ *is continuous.*

7 Absolute Continuity

Definition 13 (Bounded Variation)**.** *Let* $f : [a,b] \to \mathbb{R}$*. For a partition* $\Gamma = \{a = x_0, x_1, \ldots, x_n = b\}$ *of the interval* $[a,b]$*, we define the sum*

$$S_\Gamma(f) := \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|.$$

The function f *is said to be* Bounded Variation *if* $\sup_\Gamma S_\Gamma(f) < \infty$*.*

Definition 14 (Absolute Continuity)**.** *A function* $f : [a,b] \to \mathbb{R}$ *is* Absolutely Continuous *if for every* $\epsilon > 0$*, there exists a* $\delta > 0$ *such that for any collection of non-overlapping sub-intervals* $\{[a_i, b_i]\}_{i=1}^{\infty}$ *of* $[a,b]$ *with*

$$\sum_{i=1}^{\infty}(b_i - a_i) < \delta$$

then we have

$$\sum_{i=1}^{\infty} |f(b_i) - f(a_i)| < \epsilon.$$

Theorem 30 (Absolute Continuity $\Leftrightarrow$ Integration by Parts)**.** *A function* $f : [a,b] \to \mathbb{R}$ *is absolutely continuous if and only if* f' *exists a.e.,* f' *is integrable, and*

$$f(x) - f(a) = \int_a^x f', \qquad \text{for } a \leq x \leq b.$$

8 Functional Analysis*

Theorem 31 (Baire Category Theorem)**.** *Let* (X, d) *be a complete metric space. Then for any countable collection* $\{U_n\}_{n=1}^{\infty}$ *of open dense subsets of* X*,*

$$\bigcap_{n=1}^{\infty} U_n \text{ is dense in } X.$$

Definition 15 (Weak/Weak* Convergence)**.** *Let X be a Banach space. Let $\{f_n\} \subset X^*$ be a sequence of functions. Then we say that f_n converges weakly to $f \in X^*$ if for every $G \in X^{**}$ we have.*

$$G(f_n) \to G(f).$$

We say that f_n converges weak to $f \in X^*$ if for every $x \in X$ we have*

$$f_n(x) \to f(x).$$

If X is reflexive (in other words, $X \simeq X^{**}$) then we have weak convergence and weak convergent are equivalent.*

Theorem 32 (Hahn-Banach)**.** *...*

Theorem 33 (Uniform Boundedness Principle)**.** *...*

Theorem 34 (Open Mapping Theorem)**.** *...*

Theorem 35 (Closed Graph Theorem)**.** *...*

Theorem 36 (Banach-Alaoglu)**.** *...*