Real Analysis I

Lecture Notes

Joseph K. Miller

Last updated: June 30, 2024

Abstract: These notes were prepared for a summer "bootcamp" to prepare graduate students in mathematics for the real analysis prelim exam. The sections, theorems, and definitions marked with $*$ below will stand for more advanced topics which are less likely to appear on exams.

Contents

1 Measure Theory **2**
⌜How do we measure sets?⌟

2 Measurable Functions **4**
⌜The family of measurable functions is almost continuous.⌟

3 Integration **5**
⌜When do limits and integrals commute?⌟

4 L^p spaces **6**
⌜Quantify the distribution of mass for functions.⌟

5 Fubini's Theorem **8**
⌜Do integrals on product spaces commute?⌟

6 Lebesgue Differentiation Theorem **8**
⌜Local averages converge almost everywhere.⌟

7 Absolute Continuity **9**
⌜When is the fundamental theorem of calculus valid?⌟

8 Functional Analysis* **10**
⌜Infinte dimensional linear algebra⌟

1 Measure Theory

Definition 1 (Outer Measure)**.** *Let X be a set and $\mu : 2^X \to [0,\infty]$. We say μ is an outer measure if it is countably sub-additive and $\mu(\emptyset) = 0$.*

Definition 2 (Lebesgue Measure)**.** *The Lebesgue measure $\lambda_d : 2^{\mathbb{R}^d} \to [0,\infty]$ is defined by*

$$\lambda_d(E) := \inf\left\{\sum_{k=1}^{\infty} vol(Q_k) : \quad E \subset \bigcup_{k=1}^{\infty} Q_k, \quad Q_k \text{ cubes}\right\} \tag{1}$$

where $vol(Q)$ is the volume of the cube Q. It is an outer measure.

Definition 3 (Measure)**.** *Let X be a set and $\mathcal{F} \subset 2^X$ be a σ-algebra. A function $\mu : \mathcal{F} \to [0,\infty]$ is a measure if μ is countably additive. The sets $E \in \mathcal{F}$ are the measurable sets.*

Theorem 1 (Carethéodory's Theorem)**.** *Let X be a set and μ be an outer measure. Then the set*

$$\mathcal{F} := \{E \subset X : \forall A \subset X, \quad \mu(A) = \mu(A \cap E) + \mu(A \cap E^c)\} \tag{2}$$

is a σ-algebra, and $\mu : \mathcal{F} \to [0,\infty]$ is a measure. We therefore say $E \in \mathcal{F}$ is measurable with respect to the outer measure μ.

Theorem 2 (Lebesgue Measurability)**.** *For each $E \subset \mathbb{R}^d$, the following are equivalent*

- *For all $A \subset \mathbb{R}^d$, $\lambda_d(A) = \lambda_d(A \cap E) + \lambda_d(A \cap E^c)$.*
- *For each $\epsilon > 0$, there exists an open set $O \supset E$ such that $\lambda_d(O \setminus E) < \epsilon$.*

Theorem 3 (Measuring Tools)**.** *Let μ be a measure on X. Let A_k be measurable for $k \in \mathbb{N}$.*

- *If $A_1 \subset A_2 \subset \cdots$ and $A^* = \bigcup_{k=1}^{\infty} A_k$, then $\mu(A^*) = \lim_{k\to\infty} \mu(A_k)$.*
- *If $A_1 \supset A_2 \supset \cdots$, $A_* = \bigcap_{k=1}^{\infty} A_k$, and $\mu(A_1) < \infty$ then $\mu(A_*) = \lim_{k\to\infty} \mu(A_k)$.*
- *Let us define*

  $$\limsup_{k\to\infty} A_k := \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} A_j.$$

  If $\sum_{k=1}^{\infty} \mu(A_k) < \infty$, then $\mu(\limsup_{k\to\infty} A_k) = 0$. This implication is known as the Borel-Cantelli Lemma.

Theorem 4 (Vitali's Covering Theorem)**.** *Let $\mathcal{F}$ be a collection of non-degenerate closed balls in $\mathbb{R}^d$ that satisfies*

$$\sup\{diam(B)|B \in \mathcal{F}\} < \infty \tag{3}$$

where diam(B) is the diameter of the ball B. Then there exists a countable subfamily $\mathcal{G} \subset \mathcal{F}$ of disjoint closed balls such that

$$\bigcup_{B\in\mathcal{F}} \subset \bigcup_{B\in\mathcal{G}} 5B.$$

Theorem 5 (Open Set Approximation)**.** *Let $U \subset \mathbb{R}^d$ be open, and $R > 0$. Then, there exists a countable collection $\mathcal{G}$ of disjoint closed balls in $\mathbb{R}^d$ with $diam(B) < R$ for all $B \in \mathcal{G}$ and*

$$\lambda_d\left(U \setminus \bigcup_{B\in\mathcal{G}} B\right) = 0.$$

Definition 4 (Singular and Absolutely Continuous)**.** *Let μ, ν be two measures on X.*

- *We say μ is singular with respect to ν (written $\mu \perp \nu$) if there exists a measurable set $Z \subset X$ such that*

$$\mu(Z) = \nu(X \setminus Z) = 0.$$

- *We say μ is absolutely continuous with respect to ν (written $\mu \ll \nu$) if for all measurable $A \subset X$,*

$$\nu(A) = 0 \quad \Longrightarrow \quad \mu(A) = 0.$$

Definition 5 (Weak Limits)**.** *Let $\{\mu_n\}_{n\in\mathbb{N}}, \mu$ be finite measures on a metric space X. We say μ_n weakly converges to μ (written $\mu_n \rightharpoonup \mu$) if for every bounded continuous function $\phi : X \to \mathbb{R}$ we have*

$$\int_X \phi d\mu_n \to \int_X \phi d\mu.$$

Theorem 6 (Jordan Decompsition*)**.** *Let $\mu : 2^X \to [-\infty, \infty]$ be a signed measure. Then there exists unique positive measures μ^+, μ^- such that $\mu = \mu^+ - \mu^-$.*

Theorem 7 (Hahn Decomposition*). *Let $\mu : 2^X \to [-\infty, \infty]$ be a signed measure. Then there exists measurable sets $P, N \subset X$ such that $P \cap N = \emptyset$ and $P \cup N = X$ with the property that*

$$E \subset P \quad \Longrightarrow \quad \mu(E) \geq 0$$
$$E \subset N \quad \Longrightarrow \quad \mu(E) \leq 0.$$

We call P the positive part and N the negative part of μ. They satisfy

$$\mu^+(E) = \mu(E \cap P), \qquad \mu^-(E) = -\mu(E \cap N).$$

Theorem 8 (Lebesgue Decomposition*). *Let μ, ν be σ-finite signed measures. Then there exists unique signed measures $\mu_\perp$ and $\mu_\ll$ on X such that $\mu_\perp \perp \nu$, $\mu_\ll \ll \nu$, and*

$$\mu = \mu_\perp + \mu_\ll.$$

Theorem 9 (Radon-Nikodym*). *Let μ, ν be two signed σ-finite measures on X. If $\mu \ll \nu$, then there exists a unique measurable function f on X such that for every measurable E we have*

$$\mu(E) = \int_E f d\nu.$$

We call f the Radon-Nikodym derivative $\frac{d\mu}{d\nu}$.

2 Measurable Functions

Definition 6 (Measurability). *Let $(X, \mathcal{F}, \mu)$ be a measure space. Then $f : X \to \overline{\mathbb{R}}$ is measurable if for all $\lambda \in \mathbb{R}$,*

$$\{x \in X : f(x) \geq \lambda\} \in \mathcal{F}.$$

Definition 7 (Convergence in Measure). *Let $\{f_n\}$ and f be measurable functions defined on a measurable set $E \subset \mathbb{R}^n$. Then f_n is said to converge in measure to f if for every $\lambda > 0$ we have*

$$\lim_{k \to} |\{x \in E : |f_n(x) - f(x)| \geq \lambda\}| = 0.$$

Theorem 10 (Measurable Function $\approx$ Simple). *Let $(X, \mathcal{F}, \mu)$ be a measure space, and $f : X \to [0, \infty]$ measurable. Then there exists an increasing sequence of simple functions[1] S_k such that $f(x) = \lim_{k \to \infty} S_k(x)$ for every $x \in X$.*

[1] A simple function is a finite sum of characteristic functions of measurable sets.

Theorem 11 (Egorov). *Let $E \subset \mathbb{R}^n$ be a finite measurable set and $\{f_n\}$ a sequence of measurable functions on E which converge pointwise a.e. to a limit f. Then for all $\epsilon > 0$, there exists a set $F \subset E$ such that $|F| < \epsilon$ and f_n converges uniformly to f on $E \setminus F$.*

Theorem 12 (Lusin). *Let $E \subset \mathbb{R}^n$ be a measurable set with finite measure, and let $f : E \to \mathbb{R}$ be finite valued. Then f is measurable if and only if for every $\epsilon > 0$ there exists a set $F \subset E$ with $|F| < \epsilon$ such that $f : E \setminus F \to \mathbb{R}$ is continuous (with respect to the subspace topology).*

3 Integration

Theorem 13 (Chebyshev's Inequality). *Let $f \in L^1(E)$. Then we have for all $\alpha > 0$ that*

$$|\{x \in E : |f(x)| \geq \alpha\}| \leq \frac{1}{\alpha} \int_E |f(x)| dx.$$

Theorem 14 (Fatou's Lemma). *Let $\{f_n\}_{n=1}^\infty \subset L^1(E)$ be a sequence of positive functions. Then we have*

$$\int_E \liminf_{n\to\infty} f_n(x) dx \leq \liminf_{n\to\infty} \int_E f_n(x) dx.$$

This holds even if the left hand side is infinite.

Theorem 15 (Monotone Convergence). *Let $\{f_n\}_{n=1}^\infty \subset L^1(E)$ be an* increasing *sequence of positive functions which converge pointwise almost everywhere to f. Then, regardless of the integrability of f, we have*

$$\lim_{n\to\infty} \int_E f_n(x) dx = \int_E f(x) dx.$$

Theorem 16 (Bounded Convergence). *Let $|E| < \infty$. If $\{f_n\} \subset L^1(E)$ is a sequence of functions such that $|f_n| \leq M < \infty$ for all n, and $f_n \to f$ a.e., then*

$$\lim_{n\to\infty} \int_E f_n(x) dx = \int_E f(x) dx.$$

Theorem 17 (Dominated Convergence). *Let $\{f_n\} \subset L^1(\mathbb{R}^d)$ be a sequence of functions which converge pointwise a.e. to $f \in L^1(\mathbb{R}^d)$. If there is an integrable function $g \in L^1(\mathbb{R}^d)$ such that $|f_n| \leq |g|$ for all n, then we have*

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n(x) dx = \int_{\mathbb{R}^d} f(x) dx.$$

Theorem 18 (Jensen's Inequality)**.** *Let $f \in L^1(E)$ with $|E| < \infty$, and let $\varphi : \mathbb{R} \to \mathbb{R}$ be a* convex *function. Then we have*

$$\varphi\left(\frac{1}{|E|}\int_E f(x)dx\right) \leq \frac{1}{|E|}\int_E \varphi(f(x))dx.$$

This inequality is reversed if φ is concave.

Theorem 19 (Generalized Dominated Convergence*)**.** *Let $\{f_n\} \subset L^1(\mathbb{R}^d)$ be a sequence of functions that converge pointwise a.e. to f. Let $\{g_n\} \subset L^1(\mathbb{R}^d)$ be a sequence of functions such that*

- *$|f_n| \leq |g_n|$ for each n,*
- *There exists a $g \in L^1(\mathbb{R}^d)$ such that $g_n \to g$ a.e.*
- *We have as $n \to \infty$*
$$\int_{\mathbb{R}^d} g_n(x)dx \to \int_{\mathbb{R}^d} g(x)dx$$

Then, we obtain that

$$\int_{\mathbb{R}^d} f_n(x)dx \to \int_{\mathbb{R}^d} f(x)dx.$$

4 L^p spaces

Definition 8 (Conjugate Exponents)**.** *Let $1 \leq p \leq \infty$. We define p' to be the* conjugate exponent *to p given by the unique number which satisfies*

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

For $p = 1$, we take $p' = \infty$ and for $p = \infty$, we take $p' = 1$.

Theorem 20 (Hölder's Inequality)**.** *Let $1 \leq p \leq \infty$. If p' is the conjugate exponent to p, then for any f, g measurable we have*

$$\int |fg| \leq \|f\|_{L^p}\|g\|_{L^{p'}}$$

Theorem 21 (Minkowski's Inequality)**.** *Let f, g be measurable functions and $1 \leq p \leq \infty$. Then*

$$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$$

Theorem 22 (Reverse Hölder)**.** *Let f be a measurable function. Then for $1 \le p \le \infty$ we can calculate*

$$\|f\|_{L^p} = \sup\left\{\left|\int_{\mathbb{R}^d} f(x)g(x)dx\right| : \; g \in L^{p'} \text{ and } \|g\|_{L^{p'}} \le 1\right\}.$$

Theorem 23 (L^p Duality/Riesz Representation)**.** *Let $1 < p < \infty$. Then*

$$\left(L^p(\mathbb{R}^d)\right)^* \simeq L^{p'}(\mathbb{R}^d).$$

That is, any continuous linear map $T : L^p(\mathbb{R}^d) \to \mathbb{R}$ can be identified with a function $g \in L^{p'}(\mathbb{R}^d)$ such that

$$T(f) = \int_{\mathbb{R}^d} f(x)g(x)dx.$$

Definition 9 (Weak Convergence in L^p)**.** *For $1 < p < \infty$ we say a sequence $\{f_n\} \subset L^p$ converges weakly to $f \in L^p$, written as $f_n \rightharpoonup f$ if for every $g \in L^{p'}$ we have*

$$\int f_n g \to \int fg.$$

For $p = \infty$, we can give an identical definition, but instead we call it weak * convergence *(this is because $(L^1)^* \simeq L^\infty$, but $(L^1)^{**} \not\simeq L^1$).*

Theorem 24 (Weak Compactness in L^p)**.** *For $1 < p < \infty$, for every bounded sequence $\{f_n\} \subset L^p(\mathbb{R}^d)$, there exists a subsequence $\{f_{n_k}\}$ and $f \in L^p(\mathbb{R}^d)$ such that $f_{n_k} \rightharpoonup f$. Equivalently, this says that the unit ball on L^p is compact in the weak topology.*

Definition 10 (Convolution)**.** *For two functions $f, g \in L^1(\mathbb{R}^d)$ we define their convolution by*

$$f * g(x) := \int_{\mathbb{R}^d} f(x-y)g(y)dy.$$

One can check that this defines an L^1 function. This same formula defines convolution on different classes of spaces.

Theorem 25 (L^2 as a Hilbert Space)**.** *We can define an inner product on $L^2(\mathbb{R}^d)$ by considering*

$$\langle f, g\rangle_{L^2} := \int_{\mathbb{R}^d} f(x)\overline{g(x)}dx.$$

Definition 11 (Weak L^p)**.** *A measurable function f lies in weak L^p if*

$$N_p(f) := \sup_{\lambda > 0} \lambda^p |\{x : |f(x)| \ge \lambda\}| < \infty$$

We call the quantity $N_p(f)$ the weak L^p norm of f.

5 Fubini's Theorem

Theorem 26 (Fubini's Theorem)**.** *Let $f = f(x,y) \in L^1(\mathbb{R}^d \times \mathbb{R}^d)$. Then, for almost every x, the function $y \mapsto f(x,y)$ is measurable and in $L^1(\mathbb{R}^d)$. Moreover, we have*

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} f(x,y)d(x,y) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x,y)dxdy = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x,y)dydx.$$

Theorem 27 (Tonelli's Theorem)**.** *Let $f = f(x,y) \geq 0$ be measurable. Then for almost every x the function $y \mapsto f(x,y)$ is measurable and*

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} f(x,y)d(x,y) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x,y)dxdy = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x,y)dydx.$$

This holds regardless of the integrability of f.

Theorem 28 (Layer Cake Formula)**.** *For $1 \leq p < \infty$, we can calculate the L^p norm of a measurable function f by the formula*

$$\int_{\mathbb{R}^d} |f|^p = p \int_0^\infty \lambda^{p-1} |\{x \in \mathbb{R}^d : |f(x)| \geq \lambda\}| d\lambda$$

6 Lebesgue Differentiation Theorem

Theorem 29 (Lebesgue Differentiation Theorem)**.** *Let $f \in L^1_{loc}(\mathbb{R}^d)$. Then for almost every point $x \in \mathbb{R}^d$, we have*

$$\lim_{r \to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)|dy = |f(x)|.$$

This theorem can be generalized to

$$\lim_{r \to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - f(x)|dy = 0,$$

and for more general classes of balls B_r.

Definition 12 (Maximal Function)**.** *For $f \in L^1(\mathbb{R}^d)$, we define the Hardy-Littlewood Maximal function by*

$$\mathcal{M}f(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)|dy$$

where the supremum is taken over all balls containing x.

Theorem 30 (Vitali's Covering Lemma). *Let $\mathcal{G}$ be a possibly uncountable collection of balls $B \subset \mathbb{R}^d$ which have uniformly bounded diameter. Then there exists a countable collection of disjoint balls $\{B_k\}_{k=1}^{\infty}$ such that*

$$\bigcup_{\mathcal{G}} B_\alpha \subset \bigcup_{k=1}^{\infty} 5B_k.$$

Here, we are using the notation that $5B_k$ is the ball with the same center as B_k but with 5 times its radius.

Theorem 31 (Hardy-Littlewood Maximal Inequality). *There exists a constant $C > 0$ such that for any $f \in L^1(\mathbb{R}^d)$ we have*

$$|\{x \in \mathbb{R}^d : |\mathcal{M}f(x)| \geq \lambda\}| \leq \frac{C}{\lambda}\|f\|_{L^1(\mathbb{R}^d)}.$$

Equivalently, the maximal function $\mathcal{M} : L^1(\mathbb{R}^d) \to L^1_{weak}(\mathbb{R}^d)$ is continuous.

7 Absolute Continuity

Definition 13 (Bounded Variation). *Let $f : [a, b] \to \mathbb{R}$. For a partition $\Gamma = \{a = x_0, x_1, \ldots, x_n = b\}$ of the interval $[a, b]$, we define the sum*

$$S_\Gamma(f) := \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|.$$

The function f is said to be Bounded Variation *if $\sup_\Gamma S_\Gamma(f) < \infty$.*

Definition 14 (Absolute Continuity). *A function $f : [a, b] \to \mathbb{R}$ is* Absolutely Continuous *if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for any collection of non-overlapping sub-intervals $\{[a_i, b_i]\}_{i=1}^{\infty}$ of $[a, b]$ with*

$$\sum_{i=1}^{\infty}(b_i - a_i) < \delta$$

then we have

$$\sum_{i=1}^{\infty} |f(b_i) - f(a_i)| < \epsilon.$$

Theorem 32 (Absolute Continuity $\Leftrightarrow$ Integration by Parts). *A function $f : [a, b] \to \mathbb{R}$ is absolutely continuous if and only if f' exists a.e., f' is integrable, and*

$$f(x) - f(a) = \int_a^x f', \qquad \text{for } a \leq x \leq b.$$

8 Functional Analysis*

Theorem 33 (Baire Category Theorem). *Let (X, d) be a complete metric space. Then for any countable collection $\{U_n\}_{n=1}^{\infty}$ of open dense subsets of X,*

$$\bigcap_{n=1}^{\infty} U_n \text{ is dense in } X.$$

Definition 15 (Weak/Weak* Convergence). *Let X be a Banach space. Let $\{f_n\} \subset X^*$ be a sequence of functions. Then we say that f_n converges weakly to $f \in X^*$ if for every $G \in X^{**}$ we have.*

$$G(f_n) \to G(f).$$

We say that f_n converges weak to $f \in X^*$ if for every $x \in X$ we have*

$$f_n(x) \to f(x).$$

If X is reflexive (in other words, $X \simeq X^{**}$) then we have weak convergence and weak convergent are equivalent.*

Theorem 34 (Hahn-Banach). *...*

Theorem 35 (Uniform Boundedness Principle). *...*

Theorem 36 (Open Mapping Theorem). *...*

Theorem 37 (Closed Graph Theorem). *...*

Theorem 38 (Banach-Alaoglu). *...*